Inclusion of Fractal Geometry into Virginia 2009 *Mathematics Standards of Learning* Curriculum Framework

Examples of current mathematics standards which can be modified to add fractals to the curriculum framework include nearly every grade 3 – 12. The following examples cover the 6 major understandings of fractal geometry:

Grade 5 – The new standard 5.13 (*The student, using plane figures (square, rectangle, triangle, parallelogram, rhombus, and trapezoid), will a) develop definitions of these plane figures; and b) investigate and describe the results of combining and subdividing plane figures*.) has a perfect fit while subdividing plane figures. One type of subdivision creates self-similarity. (See Appendix A)

The new standard 5.17 (*The student will describe the relationship found in a number pattern and express the relationship*) can introduce students to fractal patterns – patterns within patterns.

Grade 6 – The new standard 6.17 (*The student will identify and extend geometric and arithmetic sequences*.) needs only a bullet in the framework to compare arithmetic, geometric, Fibonacci and fractal patterns in numbers. (See Appendix B)

Grade 7 – The new standard 7.13 (*The student will determine if plane figures — quadrilaterals and triangles — are similar and write proportions to express the relationships between corresponding parts of similar figures*) has similarity in the framework. Adding self-similarity allows for a natural application of similarity. (See Appendix C)

Grade 8 – The new standard 8.8 (*The student will a) apply transformations to plane figures; and b) identify applications of transformations*.) asks students to apply transformations. Replicating fractal patterns from nature is a real-world application which gives students an introduction to the recursive process while demonstrating self-similarity both mathematically and naturally. (See Appendix D)

Algebra 1 – Though recursion/iteration is an important mathematical concept, it is found nowhere in the standards. Standard A.7 describes what students are to know about functions. Asking a learner to iterate a function allows the student some focused practice assisting them in the assimilation of new knowledge. (See Appendix E)

Algebra 2 – Fractal dimension is used to measure roughness. One method of measurement is the Hausdorff Dimension which employs the use of slope and a log-log graph – both are important concepts in Algebra 2. Standard AII.6 on logarithms will be enriched with 21st Century skills by adding dimension calculation. (See Appendix F)

Computer Mathematics – Fractals are widely used by the computer graphics community from image compression to recreating natural scenes in movies. Standard COM.8 allows learners to design and implement computer graphics. The addition of modeling using fractals would allow students to be up to date with the latest trends from the graphics community. (See Appendix G)

Discrete Mathematics – Fractals use the recursive process to generate different sequence and series. The new framework contains a fractal standard.

STANDARD 5.13 STRAND: GEOMETRY GRADE LEVEL 5

5.13 The student, using plane figures (square, rectangle, triangle, parallelogram, rhombus, and trapezoid), will
a) develop definitions of these plane figures; and
b) investigate and describe the results of combining and subdividing plane figures.

UNDERSTANDING THE STANDARD
(Background Information for Instructor Use Only)

- A triangle is a polygon with three sides. Triangles may be classified according to the measure of their angles, i.e., right, acute, or obtuse. Triangles may also be classified according to the measure of their sides, i.e., scalene (no sides congruent), isosceles (at least two sides congruent) and equilateral (all sides congruent).
- A quadrilateral is a polygon with four sides.
- A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. Properties of a parallelogram include the following:
  – A diagonal (a segment that connects two vertices of a polygon but is not a side) divides the parallelogram into two congruent triangles.
  – The opposite sides of a parallelogram are congruent.
  – The opposite angles of a parallelogram are congruent.
  – The diagonals of a parallelogram bisect each other.
- A rectangle is a parallelogram with four right angles. Since a rectangle is a parallelogram, a rectangle has the same properties as those of a parallelogram.
- A square is a rectangle with four congruent sides. Since a square is a rectangle, a square has all the properties of a rectangle and of a parallelogram.

A plane figure is said to be self-similar if it is made up of only smaller, similar versions of itself.

ESSENTIAL UNDERSTANDINGS

All students should

- ~~Understand that the defining properties and symmetry of various plane figures are unique.~~
- Understand that simple plane figures can be combined to make more complicated figures and that complicated figures can be subdivided into simple plane figures.
- Understand that if a figure is composed of only similar figures, then it is a self-similar object.

ESSENTIAL KNOWLEDGE AND SKILLS

The student will use problem solving, mathematical communication, mathematical reasoning, connections and representation to

- ~~Recognize and identify the properties of~~ Develop definitions for squares, rectangles, triangles, parallelograms, rhombi, kites and trapezoids.
- ~~Describe the properties of squares, rectangles, triangles, parallelograms, rhombi, kites and trapezoids.~~
- ~~Analyze the properties of squares, rectangles, triangles, parallelograms, rhombi, kites and trapezoids.~~
- ~~Identify congruent, non-congruent, and similar figures.~~
- Investigate and ~~Describe~~ describe the results of combining and subdividing ~~shapes~~ plane figures.
- ~~Identify and describe a line of symmetry.~~
- ~~Recognize the images of figures resulting from geometric transformations such as translation, reflection, or rotation.~~
- Describe and find plane figures which have self-similarity.

STANDARD 5.17 **STRAND: GEOMETRY** **GRADE LEVEL 5**

5.17 The student will describe the relationship found in a number pattern and express the relationship.

UNDERSTANDING THE STANDARD
(Background Information for Instructor Use Only)

- There are an infinite number of patterns. The simplest types of patterns are repeating patterns. In such patterns, students need to identify the basic unit of the pattern and repeat it.
- Growing patterns are more difficult for students to understand than repeating patterns because not only must they determine what comes next, they must also begin the process of generalization.
- Students need experiences with growing patterns in both numerical and geometric formats.
- Sample numerical patterns are

  6, 9, 12, 15, 18, …;

  5, 7, 9, 11, 13, …;

  1, 2, 4, 7, 11, 16, …;

  2, 4, 8, 16, 32, …; ~~and~~
  32, 30, 28, 26, 24…;
  1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5,… and

  1, 5, 25, 125, 625, ….

- A fractal contains an infinite number of copies of itself. One method of determining a fractal pattern is trimming – that is, deleting the first occurrence of each number. The remaining numbers form the original pattern.

  e.g. 1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,…
  trims down to
  ~~1~~,1,~~2~~,1,2,~~3~~,1,2,3,~~4~~,1,2,3,4,~~5~~…
  1, 1,2, 1,2,3, 1,2,3,4,…

ESSENTIAL UNDERSTANDINGS

All students should

- Understand that patterns and functions can be represented in many ways and described using words, tables, graphs, and symbols.
- Understand the structure of a pattern and how it grows or changes using concrete materials and calculators.
- Understand that mathematical relationships exist in patterns including patterns in nature.
- Understand that an expression uses symbols to define a relationship.
- Understand that expressions can be numerical or variable or a combination of numbers and variables.

ESSENTIAL KNOWLEDGE AND SKILLS

The student will use problem solving, mathematical communication, mathematical reasoning, connections and representation to

- Describe numerical, fractal and geometric patterns formed by using concrete materials and calculators.
- ~~Express~~ Describe the relationship found in ~~numerical and geometric~~ patterns, using words, tables, ~~graphs, or a mathematical sentence~~ and symbols to express the relationship.

STANDARD 6.17 STRAND:PATTERNS, FUNCTIONS, AND ALGEBRA GRADE LEVEL 6

6.17 The student will identify and extend geometric and arithmetic sequences.

UNDERSTANDING THE STANDARD (Background Information for Instructor Use Only)	ESSENTIAL UNDERSTANDINGS	ESSENTIAL KNOWLEDGE AND SKILLS
• Numerical patterns may include linear and exponential growth, perfect squares, triangular and other polygonal numbers, fractal, or Fibonacci numbers. Arithmetic and geometric sequences are types of numerical patterns. • In the numerical pattern of an arithmetic sequence, students must determine the difference, called the *common difference*, between each succeeding number in order to determine what is added to each previous number to obtain the next number. Sample numerical patterns are 6, 9, 12, 15, 18, …; and 5, 7, 9, 11, 13, …. • In geometric number patterns, students must determine what each number is multiplied by to obtain the next number in the geometric sequence. • This multiplier is called the *common ratio*. Sample geometric number patterns include 2, 4, 8, 16, 32,…; 1, 5, 25, 125, 625, …; and 80, 20, 5, 1.25, … • Strategies to recognize and describe the differences between terms in numerical patterns include, but are not limited to, examining the change between consecutive terms, ~~looking for prime numbers~~, and finding common factors. An example is the pattern 1, 2, 4, 7, 11, 16,…	What is the difference between an arithmetic and a geometric sequence? While both are numerical patterns, arithmetic sequences are additive and geometric sequences are multiplicative.	• Investigate and apply strategies to recognize and describe the change between terms in numerical patterns. • Investigate and apply strategies to recognize and describe geometric patterns. • Describe verbally and in writing the relationships between consecutive terms in a numerical or geometric pattern. • Extend and apply numerical and geometric patterns to similar situations. • ~~Create~~ Using a table as an organizing tool, extend numerical and geometric ~~patterns~~ sequences by using a given rule or mathematical relationship. ~~• Describe numerical and geometric patterns, including triangular numbers.~~ • Compare and contrast arithmetic, and geometric sequences. • Identify the common difference for a given arithmetic sequence. • Identify the common ratio for a given geometric sequence.

7.6 The student will determine if plane figures – quadrilaterals and triangles – are similar and write proportions to express the relationships between corresponding sides of similar figures.

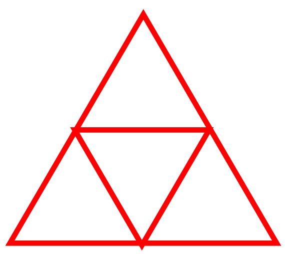
UNDERSTANDING THE STANDARD (Teacher Notes)

- Two polygons are similar if corresponding (matching) angles are congruent and the lengths of corresponding sides are proportional.
- Congruent polygons have the same size and shape.
- Congruent polygons are similar polygons for which the ratio of the corresponding sides is 1:1.
- Similarity statements can be used to determine corresponding parts of similar figures such as:

  ΔABC ≈ ΔDEF

  <A corresponds to <D
  AB corresponds to *DE*

- One of the key traits of a fractal is its self-similarity. A polygon is self-similar if it is made up only of exact copies of itself.



- The traditional notation for marking corresponding angles is to use a curve on each angle. Denote which angles correspond with the same number of curved lines. For example, if <A corresponds to <B, then both angles will be marked with the same number of curved lines.

ESSENTIAL UNDERSTANDINGS

- ~~All students should~~
- ~~Understand that similar geometric figures have the same shape but may have different sizes.~~
- ~~Understand how ratios and proportions can be used to determine the length of something that cannot be measured directly.~~
- How do polygons that are similar compare to polygons that are congruent? Congruent polygons have the same size and shape. Similar polygons have the same shape, and corresponding angles between the similar figures are congruent. However, the lengths of the corresponding sides are proportional. All congruent polygons are considered similar with the ratio of the corresponding sides being 1:1.
- How are similar and self-similar alike? How are they different?
- How are self-similar plane figures created?

ESSENTIAL KNOWLEDGE AND SKILLS

- **The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to**
- Identify corresponding sides and corresponding angles of similar figures using the traditional notation of curved lines for the angles and hatch marks on the sides.
- Write proportions to express the relationships between the lengths of corresponding sides of similar figures.
- ~~Examine congruence of corresponding angles and proportionality of corresponding sides to d~~ Determine if quadrilaterals or triangles are similar by examining congruence of corresponding angles and proportionality of corresponding sides.
- Given two similar figures, write similarity statements using symbols such as ΔABC ≈ ΔDEF , <A corresponds to <D, and AB corresponds to DE .
- Analyze plane figures for self-similarity.
- Identify similarity in a self-similar figure.

8.8 The student will
a) apply transformations to plane figures; and
b) identify applications of transformations.

Appendix D

UNDERSTANDING THE STANDARD (Teacher Notes)	**ESSENTIAL UNDERSTANDINGS**	**ESSENTIAL KNOWLEDGE AND SKILLS**
• A rotation of a geometric figure is a clockwise or counterclockwise turn of the figure around a fixed point. The point may or may not be on the figure. The fixed point is called the *center of rotation.* • A reflection of a geometric figure ~~is a flip of the figure~~ moves all of the points of the figure across an axis ~~a line~~ an axis. Each point on the reflected figure is the same distance from the line axis as the corresponding point in the original figure. • A translation of a geometric figure ~~is a slide of the figure in which~~ moves all the points on the figure ~~move~~ the same distance in the same direction. • A dilation of a geometric figure is a transformation that changes the size of a figure by a scale factor to create a similar figure. • ~~Real-life~~ Practical applications may include, but are not limited to, the following: – A rotation of the hour hand of a clock from 2:00 to 3:00 shows a turn of 30° clockwise; – A reflection of a boat in water shows an image of the boat flipped upside down with the water line being the line of reflection; – A translation of a ~~shape~~ figure on a wallpaper pattern shows the same ~~shape~~ figure slid the same distance in the same direction; and – A dilation of a model airplane is the production model of the airplane. • Any/all transformation may be used in the generation of L-system fractals which model natural fractals.	~~**All students should**~~ • ~~Understand the relationship between transformations in a coordinate plane and their application in real life.~~ • ~~Explain that rotations, reflections, translations and dilations are transformations.~~ • How does the transformation of a figure on the coordinate grid affect the congruency, orientation, location and symmetry of an image? Translations, rotations and reflections maintain congruence between the preimage and image but change location. Dilations by a scale factor other than 1 produce an image that is not congruent to the preimage but is similar. Rotations and reflections change the orientation of the image. • What is the relationship between fractals, transformations, and nature? Fractals describe the patterns found in nature. These patterns can be recreated using transformations – especially dilations.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Demonstrate the reflection of a ~~figure~~ polygon over ~~a~~ the vertical or horizontal ~~line~~ axis on a coordinate grid. • Demonstrate 90°, 180°, 270°, and 360° clockwise and counterclockwise rotations of a figure on a coordinate grid. The center of rotation will be limited to the origin. • Demonstrate the translation of a ~~figure~~ polygon on a coordinate grid. • Demonstrate the dilation of a ~~figure~~ polygon from a fixed point on a coordinate grid. • Identify practical applications of transformations including, but not limited to, tiling, fabric and wallpaper designs, art, modeling nature, and scale drawings. • Identify the type of transformation in a given example.

TOPIC: FUNCTIONS

ALGEBRA I
STANDARD A.7

Appendix E

The student will investigate and analyze function (linear and quadratic) families and their characteristics both algebraically and graphically, including

- **a) determining whether a relation is a function;**
- **b) domain and range;**
- **c) zeros of a function;**
- **d) *x*- and *y*-intercepts;**
- **e) finding the values of a function for elements in its domain; and**
- **f) making connections between and among multiple representations of functions including concrete, verbal, numeric, graphic, and algebraic.**

ESSENTIAL UNDERSTANDINGS

$[x, f(x)]$ is a member of f.

- An object x in the domain of f is an x-intercept or a zero of a function f if and only if $f(x) = 0$.
- Set builder notation may be used to represent domain and range of a relation.
- Iteration and recursion are terms which describe using the same set of rules over and over. Mathematical fractals are formed by iterating a function.

ESSENTIAL KNOWLEDGE AND SKILLS

- For each x in the domain of f, find $f(x)$.
- ~~Identify the zeros of the function algebraically and confirm them, using the graphing calculator.~~
- Detect patterns in data and represent arithmetic and geometric patterns algebraically.
- Detect departures from patterns in data.
- Iterate a function by taking the output of f and make it the input for the same function f. Repeat to iterate.

ALGEBRA II

STANDARD AII.6

The student will recognize the general shape of function (absolute value, square root, cube root, rational, polynomial, exponential, and logarithmic) families and will convert between graphic and symbolic forms of functions. A transformational approach to graphing will be employed. Graphing calculators will be used as a tool to investigate the shapes and behaviors of these functions.

ESSENTIAL UNDERSTANDINGS

- The graphs/equations for a family of functions can be determined using a transformational approach.
- Transformations of graphs include translations, reflections, and dilations.
- A parent graph is an anchor graph from which other graphs are derived with transformations.
- The Hausdorff Dimension can be calculated using a log-log graph for mathematical fractals.

ESSENTIAL KNOWLEDGE AND SKILLS

The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to

- Recognize graphs of parent functions.
- Given the equation and using a transformational approach, graph a function.
- Given the graph of a function, identify the parent function.
- Given the graph of a function, identify the transformations that map the image to the preimage in order to determine the equation of the image.
- Using a transformational approach, write the equation of a function given its graph.
- ~~Recognize graphs of parent functions for linear, quadratic, absolute value, step, and exponential functions.~~
- Given the magnification, M and number of new pieces, N, calculate the Hausdorff Dimension using a log-log graph.

Appendix G

TOPIC: PROGRAM IMPLEMENTATION

COMPUTER MATHEMATICS
STANDARD COM.8

The student will design and implement computer graphics, which will include topics appropriate for the available programming environment as well as student background. Students will use graphics as an end in itself, as an enhancement to other output, and as a vehicle for reinforcing programming techniques.

ESSENTIAL UNDERSTANDINGS

- Work with computer graphics is specific to the computer operating system.
- Image compression is greatly enhanced by fractal mathematics.
- Natural images can be recreated with little memory usage using fractal algorithms.

ESSENTIAL KNOWLEDGE AND SKILLS

- Design computer graphics.
- Implement computer graphics.
- Plot points and areas.
- Determine and set window or screen dimensions.
- Determine and set screen and background colors.
- Use box commands.
- Describe the role of graphics in the computer environment.
- Describe how fractal mathematics assists in image compression.
- Design a computer graphic based on a fractal algorithm.

Selected Resources

Barnsley, M (1988). Fractals Everywhere. Boston, MA: Harcourt, Brace, Jovanovich, Publishers.

Business Week. April 1, 2002. http://www.businessweek.com/magazine/content/02_13/c3776106.htm

Frame, M.L. & Mandelbrot, B. B. (2002). Fractals, Graphics, & Mathematics Education. Washington, DC: The Mathematical Association of America.

Introductory Website on Fractals. http://math.rice.edu/~lanius/frac/

State of New Jersey, Department of Education. http://www.nj.gov/education/ccs/

State of Virginia, Department of Education. http://www.doe.virginia.gov/